

Local Derivations on Jordan Triples

Michael Mackey*

July 24, 2012

Abstract

R.V. Kadison defined the notion of local derivation on an algebra and proved that every continuous local derivation on a von Neumann algebra is a derivation [13]. We provide the analogous result in the setting of Jordan triples.

1 Introduction

R.V. Kadison gave the definition of local derivation on an associative algebra and proved the fundamental result that every continuous local derivation on a von Neumann algebra is a derivation. In the intervening period, a substantial body of literature has been built up on the topic of local derivations and local automorphisms. As example, B.E. Johnson [12] extended Kadison's theorem to derivations on arbitrary C^* -algebras and also showed that the continuity of such local derivations is automatic.

Jordan algebras, Jordan Banach algebras and JC^* -algebras are a widely studied generalisation of their associative counterparts. These structures are further subsumed into triple analogues: Jordan triples, Jordan Banach triples and JC^* - and JB^* -triples, which can be loosely interpreted as “rectangular” versions of their binary or “square” forebears. Definitions follow below. It is our intention in this note to extend the main result of [13] to the the setting of Jordan triples. Let us make some initial limitations to our study. Firstly, while the original result of Kadison (and that of Johnson) related to module-valued maps, we deal only with self-maps on the Jordan triple. Modules are not commonly considered in Jordan triple theory, one reason (other than the algebraic difficulty) being that, frequently, a particular module over a (Jordan) algebra may be, itself, a (Jordan) triple and so passing to the triple setting obviates the need to consider modules. As examples, the right module $\mathcal{L}(H, K)$ over the C^* -algebra $\mathcal{L}(H)$ is a JB^* -triple, and every Hilbert C^* -module is a JB^* -triple [11].

Let us point out that, while Jordan triple structures generalise their binary associative counterparts, specific properties of elements or mappings may not. For example, the triple analogue of a (binary) idempotent is known as a tripotent, but an idempotent of an algebra may not be a tripotent when that algebra is viewed as a triple. Another example, most pertinent to us, is that a derivation on an algebra may not be a triple derivation when the algebra is considered as a Jordan triple. Indeed, we examine this aspect more closely in Section 3. Thus, while our main result is, in spirit, a generalisation of that of Kadison, it also provides something new in the category of von Neumann algebras.

2 Background and Terminology

For the reader unfamiliar with the notions of Jordan algebra and Jordan * -algebra we refer to [7] or [17].

2.1 Definition 1. Let Z be a complex vector space. A **TRIPLE PRODUCT** on Z is a real tri-linear map $\{\cdot, \cdot, \cdot\} : Z^3 \rightarrow Z$, $(x, y, z) \mapsto \{x, y, z\}$, which is

*UCD School of Mathematical Sciences
AMS Subject Classification (2000): 17C65, 47Cxx

email: michael.mackey@ucd.ie

- (a) complex linear in the outer variables x and z ,
- (b) symmetric in the outer variables, that is $\{x, y, z\} = \{z, y, x\}$, and
- (c) complex anti-linear in the inner variable y .

2. A JORDAN TRIPLE is a complex vector space with triple product which satisfies the Jordan triple identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}. \quad (1)$$

We say that a triple product is NON-DEGENERATE if $\{x, x, x\} = 0$ only when $x = 0$. Every algebra becomes a Jordan algebra under the Jordan product defined by $x \circ y = \frac{1}{2}(x.y + y.x)$ while every Jordan $*$ -algebra is a Jordan triple via the triple product

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*. \quad (2)$$

Combining these two facts, one sees that every $*$ -algebra is a Jordan triple via the triple product

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x). \quad (3)$$

By DERIVATION on an algebra, we mean a linear map d satisfying $d(x.y) = d(x).y + x.d(y)$. A JORDAN DERIVATION on an (associative) algebra is a linear map d for which $d(x^2) = x.d(x) + d(x).x$ (or equivalently and in terms of the Jordan product, $d(x \circ y) = x \circ d(y) + d(x) \circ y$). Also in the literature one finds the notion of JORDAN $*$ -DERIVATION on an algebra: a *real* linear map d for which $d(x^2) = x.d(x) + d(x).x^*$. Respective examples of these are $x \mapsto ax - xa$ and $x \mapsto ax - x^*a$. A derivation of a Jordan algebra is again a linear map d with $d(x \circ y) = x \circ d(y) + d(x) \circ y$. Note that a Jordan derivation on an algebra is a derivation on the associated Jordan algebra.

The ensuing concept for a Jordan triple is natural.

2.2 Definition A (JORDAN) TRIPLE DERIVATION is a linear map d on a Jordan triple satisfying

$$d\{x, y, z\} = \{dx, y, z\} + \{x, dy, z\} + \{x, y, dz\}.$$

The importance of the notion of triple derivation in the framework of Jordan triples is apparent upon noting that the Jordan triple identity (1) can be equivalently formulated thus: for all a and b , the map $a \square b - b \square a$ is a triple derivation, where $a \square b$ denotes the linear operator $x \mapsto \{a, b, x\}$. Indeed, a further reformulation in our complex setting is that for all a , $ia \square a$ is a triple derivation. Derivations of this form are known as *inner* derivations.

By a Jordan triple derivation on a Jordan $*$ -algebra or an associative $*$ -algebra, we mean with respect to the triple product as given by (2) and (3) respectively. Jordan triple derivations form a *real* linear space and are closed under the Lie bracket- that is if d_1 and d_2 are Jordan triple derivations then so is $d_1 d_2 - d_2 d_1$. We refer to [2] and [8] for greater detail concerning the structure of triple derivations on JB $*$ -triples, these forming a quite specialised class of non-degenerate Jordan triple, which we now introduce.

2.3 Definition A JB $*$ -TRIPLE is a complex Banach space and a Jordan triple on which the triple product is jointly continuous and satisfies for every element x :

- (i) $\sigma(x \square x) \geq 0$,
- (ii) $\exp(ix \square x)$ is a triple automorphism and a surjective linear isometry,
- (iii) $x \square x(x) = x^3$.

The class of JB^* -triples includes all C^* -algebras (via $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$) and also Hilbert space (via $\{x, y, z\} = \frac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x)$) and $\mathcal{L}(H, K)$ where H and K are Hilbert spaces. The category is more than a convenient envelope for better known structures however– it stands independently as a complex analytic category because of the following theorem proven by W. Kaup [14] following a body of work that can be traced back¹ to Elie Cartan’s classification of Hermitian symmetric spaces [3].

2.4 Theorem A Banach space is a JB^* -triple if, and only if, its open unit ball has a transitive group of biholomorphic mappings.

Let us mention some of the principle tools and relevant facts when working with a JB^* -triple Z . For all $x, y, z \in Z$, $\{x, y, z\} \leq xyz$ (the proof of which [6] does not follow easily from the definitions). The linear operator $B(x, y) \in \mathcal{L}(Z)$ defined by

$$B(x, y) = I - 2x \square y + Q_x Q_y$$

where $Q_x(z) = \{x, z, x\}$ occurs frequently and is known as the Bergman operator. On a C^* -algebra, the Bergman operator reduces to $B(x, y)z = (1 - xy^*)z(1 - y^*x)$. Derivations of JB^* -triples are automatically bounded [2].

An element $e \in Z$ for which $\{e, e, e\} = e$ is called a tripotent and a non-zero tripotent has norm one. For example, a tripotent of a C^* -algebra is an element v satisfying $v = vv^*v$, that is, a partial isometry. Each tripotent induces a splitting of Z , called the Peirce decomposition, into $Z = Z_1 \oplus Z_{\frac{1}{2}} \oplus Z_0$ where Z_k is the k -eigenspace of $e \square e$, with mutually orthogonal projections P_k onto the subspaces Z_k ,

$$\begin{aligned} P_1 &= Q_e Q_e, \\ P_{\frac{1}{2}} &= 2e \square e - 2Q_e Q_e, \\ P_0 &= B(e, e), \end{aligned}$$

satisfying $P_1 + P_{\frac{1}{2}} + P_0 = I$. Where the need arises, we write P_j^e rather than P_j to highlight the tripotent in question. With respect to this decomposition, the triple product behaves as follows:

$$\{Z_i, Z_j, Z_k\} \subset Z_{i-j+k}$$

where $i, j, k \in \{0, \frac{1}{2}, 1\}$ and $Z_p = \{0\}$ when $p \notin \{0, \frac{1}{2}, 1\}$. In addition, $Z_0 \square Z_1 = 0 = Z_1 \square Z_0$. The tripotent e is called MAXIMAL if $Z_0 = \{0\}$ and this is the case precisely when e is an extreme point of the unit ball of Z [15]. We point out that since the triple product is continuous, the set of tripotents forms a closed subset of Z . Two tripotents are said to be orthogonal if $e \square f = 0$ which is equivalent to saying $f \square e = 0$ or that $\{e, e, f\} = 0$. Note that the sum of two orthogonal tripotents is a tripotent.

The bidual of a JB^* -triple is also a JB^* -triple ([5]) and any JB^* -triple with a (necessarily unique) predual is called a JBW^* -triple (cf. [1]). While a JB^* -triple may not have any tripotents, a JBW^* -triple has an abundance. For example, the unit ball of a JBW^* -triple is the convex hull of its (maximal) tripotents. In addition, we have [9, Lemma 3.11]:

2.5 Proposition The set of tripotents is norm total in a JBW^* -triple. More precisely, each element in the JBW^* -triple can be approximated in norm by a finite linear combination of mutually orthogonal tripotents.

There is, as in the algebra setting, a close link between Jordan triple derivations and Jordan triple automorphisms.

2.6 Lemma Suppose d is a derivation on a JB^* -triple. Then $\exp d = \sum_{n=0}^{\infty} d^n / n!$ preserves the Jordan triple product. Conversely, if $\exp td$ is a triple homomorphism for all $t > 0$. Then d is a derivation.

¹We mention here the names of Jordan, von Neumann, Wigner, Tits, Kantor, Koecher, Loos, and Harris.

There is another quite particular reason why, in the context of JB^* -triples, derivations are of special interest. As proven by Kaup [14], the Jordan triple automorphisms, that is the bijective bounded linear maps which preserve the triple product, coincide precisely with the surjective linear isometries. That is, for $T \in GL(Z)$, $Tx = x$ for all $x \in Z$ if and only if $T\{x, y, z\} = \{Tx, Ty, Tz\}$ for all $x, y, z \in Z$. (When the surjectivity requirement is dropped, the situation is rather more complicated however, cf. [4].) In light of Lemma 2.6 therefore, it is valid to think of a triple derivation as the “infinitesimal” form of an isometry. Also worthy of note here is the fact that, on each of the irreducible JB^* -triples known as the Cartan Factors (see e.g. [6]), the group of inner automorphisms (that is, those in the group generated by exponentials of inner derivations) acts transitively on the manifold of tripotents of a given rank [10]. (The rank of a tripotent e is the maximum number of mutually orthogonal tripotents whose sum equals e .)

3 Jordan derivations and Jordan triple derivations

We are interested in clarifying how Jordan triple derivations relate to Jordan derivations. In particular, what algebraic conditions on a linear map $d : A \rightarrow A$ (where A is a $*$ -algebra) are equivalent to d being a Jordan triple derivation.

The reason behind this question is that neither Jordan derivations, $*$ -preserving Jordan derivations nor Jordan $*$ -derivations on an associative $*$ -algebra A produce triple derivations on the associated Jordan triple. This contrasts with, say, Jordan triple automorphisms, which certainly generalise associative $*$ -automorphisms. In particular, notice that if one assumes the $*$ -algebra A has an identity 1 (as we will throughout since any derivation on the non-unital algebra A extends to the unitisation by linearity and $d(1) := 0$) then a Jordan derivation, or Jordan $*$ -derivation on A sends the identity to 0. However the Jordan triple derivation $ia \square a$ is generally non-zero, as one can see in the definition of a JB^* -triple.

Let us begin with a simple observation.

3.1 Lemma Let δ be a Jordan triple derivation on a (unital) Jordan $*$ -algebra. Then

$$(a) \quad \delta(a \circ b) = \delta a \circ b + a \circ \delta b + \{a, \delta 1, b\},$$

$$(b) \quad \delta(b^*) = 2\delta 1 \circ b^* + (\delta b)^*.$$

Proof. For the first part, write $a \circ b = \{a, 1, b\}$ and apply δ . For the second, use $b^* = \{1, b, 1\}$. \square

We use L_x to denote the multiplication operator on a Jordan algebra, that is $L_x(y) = x \circ y$.

3.2 Lemma Let A be a unital Jordan $*$ -algebra. If $x = -x^*$ then L_x is a Jordan triple derivation. Further, the converse holds if the triple product is non-degenerate.

Proof.

$$\begin{aligned} L_x(\{a, b, c\}) &= x \circ \{a, b, c\} \\ &= \{x, 1, \{a, b, c\}\} \\ &= \{\{x, 1, a\}, b, c\} - \{a, \{1, x, b\}, c\} + \{a, b, \{x, 1, c\}\} \\ &= \{L_x a, b, c\} + \{a, b, L_x c\} - \{a, \{1, x, b\}, c\}. \end{aligned} \tag{4}$$

Since $\{1, x, b\} = x^* \circ b = -x \circ b = -L_x b$ we have that L_x is a Jordan triple derivation as required.

Towards the converse, we see from (4) that if L_x is a triple derivation then for all a, b and c , $\{a, \{x, 1, b\}, c\} = -\{a, \{1, x, b\}, c\}$. The non-degeneracy of the triple product implies that $x \square 1 = -1 \square x$ and in particular that $x^* = \{1, x, 1\} = -\{x, 1, 1\} = -x$. \square

The following corollary of this fact appears in [8, Lemma 1].

3.3 Corollary If δ is a triple derivation on a Jordan $*$ -algebra then so is $L_{\delta 1}$.

Proof. This follows from the fact that $\delta 1 = \delta\{1, 1, 1\} = 2\{\delta 1, 1, 1\} + \{1, \delta 1, 1\} = 2\delta 1 + (\delta 1)^*$. Thus $(\delta 1)^* = -\delta 1$ and we can apply the previous result. \square

3.4 Theorem A linear map δ on a Jordan $*$ -algebra is a Jordan triple derivation if, and only if,

(i) $\delta(a \circ b) = \delta a \circ b + a \circ \delta b + \{a, \delta 1, b\}$ for all a and b , and

(ii) $\delta(b^*) = 2\delta 1 \circ b^* + (\delta b)^*$ for all b .

Proof. The necessity of the two conditions is provided by 3.1. So suppose δ satisfies the conditions above. From (2) we have

$$\delta\{a, b, c\} = \delta((a \circ b^*) \circ c) + \delta((c \circ b^*) \circ a) - \delta((a \circ c) \circ b^*).$$

By use of (i), we expand as follows.

$$\begin{aligned} \delta((a \circ b^*) \circ c) &= \delta(a \circ b^*) \circ c + (a \circ b^*) \circ \delta c + \{a \circ b^*, \delta 1, c\} \\ &= [\delta a \circ b^* + a \circ \delta(b^*) + \{a, \delta 1, b^*\}] \circ c + (a \circ b^*) \circ \delta c + \{a \circ b^*, \delta 1, c\} \\ &= (\delta a \circ b^*) \circ c + (a \circ \delta(b^*)) \circ c + \{a, \delta 1, b^*\} \circ c + (a \circ b^*) \circ \delta c + \{a \circ b^*, \delta 1, c\} \end{aligned}$$

Similarly we have

$$\delta((c \circ b^*) \circ a) = (\delta c \circ b^*) \circ a + (c \circ \delta(b^*)) \circ a + \{c, \delta 1, b^*\} \circ a + (c \circ b^*) \circ \delta a + \{c \circ b^*, \delta 1, a\}$$

and

$$\delta((c \circ a) \circ b^*) = (\delta c \circ a) \circ b^* + (c \circ \delta a) \circ b^* + \{c, \delta 1, a\} \circ b^* + (c \circ a) \circ \delta b^* + \{c \circ a, \delta 1, b^*\}.$$

Thus

$$\begin{aligned} \delta\{a, b, c\} &= (\delta a \circ b^*) \circ c + (a \circ \delta(b^*)) \circ c + \{a, \delta 1, b^*\} \circ c + (a \circ b^*) \circ \delta c + \{a \circ b^*, \delta 1, c\} \\ &\quad + (\delta c \circ b^*) \circ a + (c \circ \delta(b^*)) \circ a + \{c, \delta 1, b^*\} \circ a + (c \circ b^*) \circ \delta a + \{c \circ b^*, \delta 1, a\} \\ &\quad - [(\delta c \circ a) \circ b^* + (c \circ \delta a) \circ b^* + \{c, \delta 1, a\} \circ b^* + (c \circ a) \circ \delta b^* + \{c \circ a, \delta 1, b^*\}] \\ &= \{\delta a, b, c\} + \{a, (\delta(b^*))^*, c\} + \{a, b, \delta c\} \\ &\quad + \{a, \delta 1, b^*\} \circ c + \{a \circ b^*, \delta 1, c\} + \{c, \delta 1, b^*\}^* \circ a + \{c \circ b^*, \delta 1, a\} \\ &\quad - \{c, \delta 1, a\} \circ b^* - \{c \circ a, \delta 1, b^*\}. \end{aligned}$$

From condition (ii), $(\delta(b^*))^* = \delta b + 2b \circ (\delta 1)^*$ and (on taking $b = 1$) $(\delta 1)^* = -\delta 1$. Therefore $(\delta(b^*))^* = \delta b - 2b \circ \delta 1$ which we use with the above to write

$$\begin{aligned} \delta\{a, b, c\} &= \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\} \\ &\quad - 2\{a, b \circ \delta 1, c\} + \left[\{a, \delta 1, b^*\} \circ c + \{a \circ b^*, \delta 1, c\} + \{c, \delta 1, b^*\}^* \circ a + \{c \circ b^*, \delta 1, a\} \right. \\ &\quad \left. - \{c, \delta 1, a\} \circ b^* - \{c \circ a, \delta 1, b^*\} \right]. \end{aligned} \tag{5}$$

Consider the square-bracketed terms in this expression. Expanding via the algebra product we have

$$\begin{aligned} \{a, \delta 1, b^*\} \circ c &= ((a \circ (\delta 1)^*) \circ b^* + (b^* \circ (\delta 1)^*) \circ a - (a \circ b^*) \circ (\delta 1)^*) \circ c \\ \{a \circ b^*, \delta 1, c\} &= ((a \circ b^*) \circ (\delta 1)^*) \circ c + (c \circ (\delta 1)^*) \circ (a \circ b^*) - ((a \circ b^*) \circ c) \circ (\delta 1)^* \\ \{c, \delta 1, b^*\} \circ a &= ((c \circ (\delta 1)^*) \circ b^* + (b^* \circ (\delta 1)^*) \circ c - (c \circ b^*) \circ (\delta 1)^*) \circ a \\ \{c \circ b^*, \delta 1, a\} &= ((c \circ b^*) \circ (\delta 1)^*) \circ a + (a \circ (\delta 1)^*) \circ (c \circ b^*) - ((c \circ b^*) \circ a) \circ (\delta 1)^* \\ -\{c, \delta 1, a\} \circ b^* &= -b^* \circ ((c \circ (\delta 1)^*) \circ a + (a \circ (\delta 1)^*) \circ c - (a \circ c) \circ (\delta 1)^*) \\ -\{c \circ a, \delta 1, b^*\} &= -((c \circ a) \circ (\delta 1)^*) \circ b^* - (b^* \circ (\delta 1)^*) \circ (c \circ a) + ((c \circ a) \circ b^*) \circ (\delta 1)^* \end{aligned}$$

Summing these, we see the square-bracketed terms in (5) can be written as

$$\{a \circ (\delta 1)^*, b, c\} + \{a, b, c \circ (\delta 1)^*\} + \{a, b \circ \delta 1, c\} - \{a, b, c\} \circ (\delta 1)^*.$$

Replacing $(\delta 1)^*$ by $-\delta 1$, we substitute into (5) to get

$$\begin{aligned} \delta\{a, b, c\} &= \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\} \\ &\quad - 2\{a, b \circ \delta 1, c\} - \{a \circ \delta 1, b, c\} - \{a, b, c \circ \delta 1\} + \{a, b \circ \delta 1, c\} + \{a, b, c\} \circ \delta 1 \\ &= \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\} \\ &\quad - \{a \circ \delta 1, b, c\} - \{a, b, c \circ \delta 1\} - \{a, b \circ \delta 1, c\} + \{a, b, c\} \circ \delta 1. \end{aligned}$$

Again since $(\delta 1)^* = -\delta 1$, Lemma 3.2 gives that $L_{\delta 1}$ is a Jordan triple derivation and so the above reduces to

$$\delta\{a, b, c\} = \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\}$$

as required. \square

3.5 Example Let A be an associative $*$ -algebra with the usual Jordan binary and triple products. For $\delta = M_{a,b}$ defined by

$$M_{a,b}(x) = xa + bx$$

condition (i) is equivalent to $(a+b)^* = -(a+b)$ while condition (ii) is equivalent to $x(a+a^*) + (b+b^*)x = 0$ for all $x \in A$.

4 Derivation Pairs

Recall that a Jordan triple isomorphism λ , is an invertible linear map which preserves the triple product, $\lambda\{x, y, z\} = \{\lambda x, \lambda y, \lambda z\}$. A more general concept is that of structure map, which is actually a pair of invertible linear maps (S, T) which satisfy $S\{x, Ty, z\} = \{Sx, y, Sz\}$ and $T\{x, Sy, z\} = \{Tx, y, Tz\}$, or equivalently,

$$S\{x, y, z\} = \{Sx, T^{-1}y, Sz\} \quad \text{and} \quad T\{x, y, z\} = \{Tx, S^{-1}y, Tx\}.$$

Structure maps can be used to define homotopes of Jordan structures [16]. A structure map (S, T) is a Jordan triple isomorphism when $S = T^{-1}$. Lemma 2.6 prompts us to the following definition.

4.1 Definition A DERIVATION PAIR on a Jordan triple is a pair of linear maps $D = (d_+, d_-)$ which satisfy

$$\begin{aligned} d_+\{x, y, z\} &= \{d_+x, y, z\} + \{x, d_-y, z\} + \{x, y, d_+z\} \\ d_-\{x, y, z\} &= \{d_-x, y, z\} + \{x, d_+y, z\} + \{x, y, d_-z\} \end{aligned}$$

for all x, y and z .

For example, the Jordan triple identity states that $(x \square y, -y \square x)$ is a derivation pair, and that $(ix \square x, ix \square x)$ is a derivation pair. Clearly d is a derivation if, and only if, (d, d) is a derivation pair. As long as the triple under question is non-degenerate, d_+ and d_- uniquely determine one another when (d_+, d_-) is a derivation pair. If (d_+^1, d_-^1) and (d_+^2, d_-^2) are derivation pairs then so is $([d_+^1, d_+^2], [d_-^1, d_-^2])$. Iterative action of the derivation pair (d_+, d_-) follows expected rules:

$$d_{\pm}^n\{x, y, z\} = \sum_{k=0}^n \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} \{d_{\pm}^k x, d_{\mp}^l y, d_{\pm}^{n-k-l} z\}.$$

Notational modifications of Lemma 2.6 guarantee the following fact.

4.2 Lemma If (d_+, d_-) is a derivation pair then $(\exp d_+, (\exp d_-)^{-1})$ is a structure map.

In fact, all the results from Section 2 can be stated in terms of derivation pairs.

4.3 Lemma Let $\delta = (\delta_+, \delta_-)$ be a Jordan triple derivation pair on a Jordan $*$ -algebra. Then

$$(a) \quad \delta_{\pm}(a \circ b) = \delta_{\pm}a \circ b + a \circ \delta_{\mp}b + \{a, \delta_{\mp}1, b\},$$

$$(b) \quad \delta_{\pm}(b^*) = 2\delta_{\pm}1 \circ b^* + (\delta_{\mp}b)^*.$$

4.4 Lemma Let A be a unital Jordan $*$ -algebra. For any $x \in A$, the pair (L_x, L_{-x^*}) is a Jordan triple derivation pair. Further, if the triple product is non-degenerate then (L_x, L_y) is a derivation pair only if $y = -x^*$.

4.5 Corollary If (δ_+, δ_-) is a triple derivation on a Jordan $*$ -algebra then so is $(L_{\delta_+1}, L_{\delta_-1})$.

4.6 Theorem A pair of linear maps (δ_+, δ_-) on a Jordan $*$ -algebra is a Jordan triple derivation pair if, and only if,

$$(i) \quad \delta_{\pm}(a \circ b) = \delta_{\pm}a \circ b + a \circ \delta_{\mp}b + \{a, \delta_{\mp}1, b\} \text{ for all } a \text{ and } b, \text{ and}$$

$$(ii) \quad \delta_{\pm}(b^*) = 2\delta_{\pm}1 \circ b^* + (\delta_{\mp}b)^* \text{ for all } b.$$

5 Local triple derivations

In [13], Kadison defines a local derivation to be a linear map which at each point a takes the same value as some derivation. This would appear to be a significant generalisation of derivation (an example, due to C. Jensen, of a local derivation which is not a derivation is given) but Kadison proceeds to show that, in the setting of von Neumann algebras at least, the definition is void:

5.1 Theorem ([13], Thm A) Every continuous local derivation on a von Neumann algebra is a derivation.

The proof is rather lengthy and ingenious. The result has the following immediate corollary, notable enough to be labelled as a theorem.

5.2 Theorem ([13], Thm B) If δ is a norm continuous linear mapping of a von Neumann algebra into itself such that for each $a \in A$ there exists $x_a \in A$ with $\delta(a) = [a, x_a]$ then there exists $x \in A$ with $\delta = [\cdot, x]$.

Our aim in this section is to introduce a similar definition of *local triple derivation* and to prove the analogous result that every continuous local triple derivation on a JBW $*$ -triple is a triple derivation. An analogue of 5.2 will ensue. It is only fair to point out that, as we have seen earlier, Jordan triple derivations are not generalisations of algebra derivations, and so our result runs parallel to Kadison's rather than being a generalisation of it. Even so, it is perhaps surprising that our proof has little in common the binary case and is substantially more compact. We attribute this to the elegant symmetry of the Jordan setting rather than any fresh ingenuity.

5.3 Definition A LOCAL TRIPLE DERIVATION on a Jordan triple Z is a linear map $\delta : Z \rightarrow Z$ such that, for every $x \in Z$, there exists a derivation δ_x with $\delta_x(x) = \delta(x)$.

Of course, every triple derivation is a local triple derivation and a natural question is whether there exist local derivations which are not derivations. The following example, a variation of one attributed to C.U. Jensen in [13], shows that such maps do exist in a purely algebraic setting.

5.4 Example Consider the $*$ -algebra $\mathbb{C}(x)$ of rational functions in the variable x over \mathbb{C} and its $*$ -subalgebra of polynomials $\mathbb{C}[x]$. This algebra, as pointed out in [13], provides an example of a local derivation which is not a derivation. The derivations of $\mathbb{C}(x)$ take the form $d_g := f \mapsto gf'$ for $g \in \mathbb{C}(x)$. Note that since the algebra is commutative, derivations and Jordan derivations agree. We are interested in triple derivations and we make the following remarks:

- (i) Either by direct calculation, or appealing to Theorem 3.4, one finds that a linear map δ on $\mathbb{C}(x)$ is a triple derivation if $\delta = \delta_{u,v}$ where $\delta_{u,v}f = uf' + ivf$ for $f \in \mathbb{C}(x)$ and u, v are self-adjoint elements of $\mathbb{C}(x)$.
- (ii) All triple derivations of $\mathbb{C}(x)$ are of the form $\delta_{u,v}$. To see this, let δ be a triple derivation. By Theorem 3.4, $(\delta 1)^* = -\delta 1$ and thus $\delta 1 = iv$ for some $v = v^*$. Also by 3.4 since $x = x^*$, $\delta x = 2(\delta 1)x + (\delta x)^*$. Thus, if we let $u = \delta x - x\delta 1$ then $u^* = u$. Now, one more use of 3.4(a) shows that

$$\begin{aligned}\delta(x^2) &= 2x\delta x + \{x, \delta 1, x\} \\ &= 2x\delta x - x^2\delta 1 \\ &= 2x(\delta x - x\delta 1) + x^2\delta 1 \\ &= u(x^2)' + iv(x^2).\end{aligned}$$

It is but a short step to showing $\delta(x^n) = u(x^n)' + iv(x^n)$ and hence $\delta(p) = up' + ivp$ for all $p \in \mathbb{C}[x]$. The extension to all $f \in \mathbb{C}(x)$ follows.

- (iii) The *local* triple derivations are the linear maps which take 1 to iv where $v = v^* \in \mathbb{C}(x)$. Indeed, if α is a local triple derivation then $\alpha(1) = \delta_{u,v}(1)$ for some self-adjoint u and v . But $\delta_{u,v}(1) = iv$. Conversely, if α is linear with $\alpha(1) = iv$ then α agrees with the derivation $\delta_{0,v}$ at any constant. Solving the functional equation $uf' + ivf = \alpha(f)$ for self-adjoint functions u and v leads to

$$u = \frac{\psi_1\phi_1 + \psi_2\phi_2}{\phi_1'\phi_1 + \phi_2'\phi_2}, \quad v = \frac{\psi_2\phi_1' - \psi_1\phi_2}{\phi_1'\phi_1 + \phi_2'\phi_2}$$

where $\alpha(f) = \psi_1 + i\psi_2$ and $f = \phi_1 + i\phi_2$ and these solutions exist in $\mathbb{C}(x)$ as long as $(\phi_1^2 + \phi_2^2)'$ is not zero, that is, when ff^* (and consequently f) is not constant. Thus $\alpha(f) = d_f(f)$ for some derivation d_f whether f is constant or not, and so α is a local derivation.

- (iv) The linear map $f \mapsto i(xf)'$, which can be written here as $\delta_{ix,1}$, is now seen to be a local derivation ($1 \mapsto i1$) but not a derivation ($ix \neq (ix)^*$).

Having seen that there do exist local triple derivations which are not derivations, let us proceed to show that no such examples exist in any JBW $*$ -triple.

5.5 Lemma Let e and f be orthogonal tripotents and δ a local derivation on a Jordan triple. Then

$$\delta\{f, e, f\} = 2\{\delta(f), e, f\} + \{f, \delta e, f\}.$$

Proof. As $e \square f = 0$ one need only show that $\{f, \delta e, f\} = 0$. For this, choose a derivation δ_e with $\delta e = \delta_e e$. Since $\delta_e\{f, e, f\} = 2\{\delta_e(f), e, f\} + \{f, \delta_e e, f\}$, we may again ignore zero products to conclude $0 = \{f, \delta_e e, f\}$ which gives the result. \square

We seek a similar result where the triple product is of the form $\{f, f, e\}$.

5.6 Lemma Let e and f be orthogonal tripotents in a Jordan triple Z and δ a local derivation on Z . Then $\{e, \delta e, f\} + \{e, e, \delta f\} = 0$.

Proof. For any tripotent e and local derivation δ , we have $\delta(e) = \delta_e(e) = \delta_e\{e, e, e\} = 2\{\delta_e e, e, e\} + \{e, \delta_e e, e\} = 2\{\delta e, e, e\} + \{e, \delta e, e\}$. Now, if e and f are orthogonal tripotents then $e + f$ is also a tripotent and so

$$\delta(e + f) = 2\{\delta(e + f), e + f, e + f\} + \{e + f, \delta(e + f), e + f\}.$$

This leads to

$$2\{\delta e, f, f\} + 2\{\delta f, e, e\} + 2\{e, \delta e, f\} + \{e, \delta f, e\} + 2\{e, \delta f, f\} + \{f, \delta e, f\} = 0$$

and, after replacing e by $-e$ and summing, we conclude

$$2\{\delta f, e, e\} + 2\{e, \delta e, f\} + \{e, \delta f, e\} = 0.$$

From the proof of Lemma 5.5, $\{e, \delta f, e\} = 0$ and thus

$$\{\delta f, e, e\} + \{e, \delta e, f\} = 0$$

as asserted. □

5.7 Corollary Let e and f be orthogonal tripotents on a Jordan triple Z and δ a local derivation on Z . Then

$$\delta\{e, e, f\} = \{\delta e, e, f\} + \{e, \delta e, f\} + \{e, e, \delta f\}.$$

5.8 Lemma Let e, f and g be mutually orthogonal tripotents on a Jordan triple Z and δ a local derivation on Z . Then

$$\delta\{e, f, g\} = \{\delta e, f, g\} + \{e, \delta f, g\} + \{e, f, \delta g\}.$$

Proof. Excluding the zero products, one must only show that $\{e, \delta f, g\} = 0$. For this, one need only choose a derivation δ_f such that $\delta_f(f) = \delta f$ and remark that $\{e, \delta_f f, g\} = 0$ since the desired identity holds for a derivation. □

At this point, we have effectively considered a number of different cases which are covered by the following proposition.

5.9 Proposition Let δ be a local triple derivation on a Jordan triple Z , Λ a family of orthogonal tripotents and e, f and g elements of Λ (not necessarily distinct). Then

$$\delta\{e, f, g\} = \{\delta e, f, g\} + \{e, \delta f, g\} + \{e, f, \delta g\}.$$

Proof. If $e = f = g$ then the result follows on replacing δ by a derivation δ_e with $\delta_e(e) = \delta e$. If e, f and g are all distinct then the result holds by Lemma 5.8. If, on the other hand, there are just two distinct tripotents, then our triple product is either of the form $\{e, f, e\}$ or $\{e, e, f\}$ and the conclusion is reached on appeal to 5.5 or 5.7 as appropriate. □

This can be further extended via the linearity of our local derivation.

5.10 Corollary Let Λ be a family of orthogonal tripotents and x, y and z elements of $\text{span } \Lambda$. Then

$$\delta\{x, y, z\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\}.$$

We can now present our main result.

5.11 Theorem Let δ be a continuous local triple derivation on a JBW*-triple. Then δ is a triple derivation.

Proof. Fix an element x and choose $\epsilon \in (0, 1)$. Then, by Proposition 2.5, we can find mutually orthogonal tripotents $\{e_k : k = 0, \dots, n\}$, a linear combination of which, say $\xi = \sum_{k=0}^n \alpha_k e_k$, has the property that $x - \xi < \epsilon$. Corollary 5.10 asserts that $\delta\{\xi, \xi, \xi\} = 2\{\delta\xi, \xi, \xi\} + \{\xi, \delta\xi, \xi\}$. It follows from $\{y, z, w\} \leq yzw$ that $\{x, x, x\} - \{\xi, \xi, \xi\} \leq C_1 x - \xi < C_1 \epsilon$ and thus by continuity of δ that $\delta(\{x, x, x\}) - \delta(\{\xi, \xi, \xi\}) \leq C_2 \epsilon$. In a similar vein, $\{\delta x, x, x\} - \{\delta\xi, \xi, \xi\} \leq C_3 \epsilon$ and $\{x, \delta x, x\} - \{\xi, \delta\xi, \xi\} \leq C_4 \epsilon$ where each C_j is positive and independent of ϵ . Let us now estimate as follows:

$$\begin{aligned} \delta\{x, x, x\} - (2\{\delta x, x, x\} + \{x, \delta x, x\}) &\leq \delta\{x, x, x\} - \delta\{\xi, \xi, \xi\} \\ &\quad + \delta\{\xi, \xi, \xi\} - (2\{\delta\xi, \xi, \xi\} + \{\xi, \delta\xi, \xi\}) \\ &\quad + 2\{\delta\xi, \xi, \xi\} - 2\{\delta x, x, x\} \\ &\quad + \{\xi, \delta\xi, \xi\} - \{x, \delta x, x\} \\ &\leq C_2 \epsilon + 0 + 2C_3 \epsilon + C_4 \epsilon. \end{aligned}$$

As ϵ is arbitrarily small, we can conclude that

$$\delta\{x, x, x\} = 2\{\delta x, x, x\} + \{x, \delta x, x\}. \quad (6)$$

The final step is a polarisation exercise. Replacing x first by $x + y$, then by $x - y$ in (6) and summing leads to

$$\begin{aligned} \delta\{x, y, y\} + \delta\{y, x, y\} + \delta\{y, y, x\} &= 2(\{\delta x, y, y\} + \{\delta y, x, y\} + \{\delta y, y, x\}) \\ &\quad + \{x, \delta y, y\} + \{y, \delta x, y\} + \{y, \delta y, x\}. \end{aligned} \quad (7)$$

In (7), replace x by ix and compare with (7) multiplied by i to gain

$$\delta\{y, x, y\} = 2\{\delta y, x, y\} + \{y, \delta x, y\}. \quad (8)$$

Finally, replace y by $y + z$ in (8) to conclude that $\delta\{y, x, z\} = \{\delta y, x, z\} + \{y, \delta x, z\} + \{y, x, \delta z\}$ and δ is a derivation. \square

Notice that continuity of the local derivation was used in quite a weak sense in this proof. If each element of the JBW*-triple is represented by a finite linear combination of orthogonal tripotents, then the conclusion remains true without appealing to the continuity of the local derivation. This is the case in any finite rank JBW*-triple and so we have the following.

5.12 Theorem Let δ be a local derivation on a finite rank JBW*-triple. Then δ is a derivation (and hence is continuous).

As an application of Theorem 5.11, we present the following.

5.13 Corollary Suppose r is a bounded linear map on a von Neumann algebra A and for every $x \in A$ there exists $p_x \geq 0$ such that $r(x) = xp_x + p_x x$. Then there exists $p \geq 0$ such that for every $x \in A$,

$$r(x^2) = xr(x) + r(x)x - xp_x x.$$

Proof. Writing $p_x = h_x^2$ for $h_x = h_x^*$ we see that $ir(x) = i(xh_x^*h_x + h_x h_x^*x) = ih_x \square h_x x$. That is, ir agrees with a triple derivation at every point and so is a local triple derivation. Therefore, by Theorem 5.11, ir is a triple derivation and, in particular,

$$\begin{aligned} ir(x^2) &= ir\{x, 1, x\} = 2\{irx, 1, x\} + \{x, ir(1), x\} \\ &= 2i(rx \circ x) + x(ip)^*x \\ &= i(x \cdot rx + rx \cdot x) - ixpx \end{aligned}$$

where $p = r(1) \geq 0$. \square

5.14 Corollary Suppose s is a bounded linear map on a von Neumann algebra A and for every $x \in A$ there exists $p_x \geq 0$ such that either $(s(x), s(x^*)) = (xp_x, x^*p_x)$ or $(s(x), s(x^*)) = (p_x x, p_x x^*)$. Then there exists $p \geq 0$ such that for every $x \in A$,

$$s(x^2) = xs(x) + s(x)x - xp_x x.$$

Proof. Defining $s^*(x) = (s(x^*))^*$ we have s is a bounded linear map and $(s + s^*)(x) = p_x x + xp_x$. Therefore, we can apply the previous corollary to $s + s^*$ to gain

$$(s + s^*)(x^2) = x(s + s^*)(x) + (s + s^*)(x)x - xp_x x.$$

Also, $(s - s^*)(x) = \pm(xp_x - p_x x)$ which means that, at x , $s - s^*$ agrees with an inner derivation. By Kadison's original result, $s - s^*$ is a derivation. Thus

$$(s - s^*)(x^2) = x(s - s^*)(x) + (s - s^*)(x)x.$$

Summing we see $s(x^2) = xs(x) + s(x)x - xp_x x$ as required. \square

We can go further in this direction. Notational changes are enough to extend Theorem 5.11 to the following variant for derivation pairs. We call a pair of linear maps (δ^+, δ^-) a *local derivation pair* if for every x there exists a derivation pair (d_x^+, d_x^-) such that $\delta^+(x) = d_x^+(x)$ and $\delta^-(x) = d_x^-(x)$.

5.15 Theorem Every local derivation pair on a JBW^* -triple is a derivation pair.

6 Concluding remarks and open questions

B.E. Johnson [12] provided the following strong extension of Kadison's theorem:

6.1 Theorem Every local derivation on a C^* -algebra is a derivation.

Notice that, apart from widening the class of algebras dealt with, Johnson's result drops the requirement of continuity of the local derivation. In particular, the automatic continuity of local C^* -algebra derivations follows from the automatic continuity of C^* -algebra derivations. This raises natural conjectures in the triple setting.

6.2 Conjecture (C1) A local triple derivation on a JB^* -triple is a derivation.

(C2) A local triple derivation on a JB^* -triple is continuous.

(C3) A continuous local triple derivation on a JB^* -triple is a derivation.

Clearly (C1) (or rather, proof thereof) implies (C3) and, by the automatic continuity of derivations [2], (C1) also implies (C2). Conversely, (C2) and (C3) together imply (C1). Johnson's proof of 6.1 relies on use of the multiplier algebra of a C^* -algebra.

The reader will bear in mind that the results of Kadison and Johnson were proven for module-valued derivations on von Neumann and C^* -algebras respectively, while in this paper we have restricted to consideration of triple derivations of the Jordan triple into itself. Recently, Peralta and Russo [18] have initiated a study of module-valued Jordan triple derivations. In particular, they answer the question of when a (module-valued) Jordan triple derivation is automatically continuous.

Remark. A triple automorphism on a Jordan triple is a bijective linear map T satisfying $T\{x, y, z\} = \{Tx, Ty, Tz\}$. The ensuing definition of local triple automorphism, analogous to Definition 5.3, is clear: a linear map T is a local triple automorphism if, for every x , there exists a triple automorphism T_x such that $T(x) = T_x(x)$. For any JB^* -triple, the set of triple automorphisms and the set of surjective linear isometries coincide. Thus, if T is a local triple automorphism then it is an isometry since, for each x , $Tx = T_x(x) = x$. In particular, we see immediately that (a) any local triple automorphism is continuous and (b) any surjective local triple automorphism is a surjective linear isometry and hence a triple automorphism.

Acknowledgement. The author thanks R.V. Hügli for helpful comments.

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